Geometric quantum dynamics with applications

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Quantum mechanics in its standard formulation is a linear and algebraic theory. In this formulation quantum states are vectors in a complex separable Hilbert space $\mathcal{H}$, the observables are represented by hermitian/self-adjoint linear operators on $\mathcal{H}$, and the dynamics of quantum systems is governed by a family of unitary operators. However, the geometric formulation of quantum mechanics sought to give a unified picture of physical systems based on its underling geometrical structures, e.g., now, the states are represented by points of a symplectic manifold with a compatible Riemannian metric, the observable are real-valued functions on the manifold, and quantum evolution is governed by the symplectic flow that is generated by a Hamiltonian function. A quantum system prepared in a pure state can be modeled on a projective Hilbert space $\mathcal{PH}$ equipped with a hermitian metric. The real and imaginary parts of this metric equips the projective Hilbert space with Riemannian and symplectic structures, respectively. Recently there have been some progress toward geometrization of quantum mechanics for pure states. However, a geometric formulation of mixed quantum states represented by density operators needs further effort to unravel its hidden properties and applications. In this line of research, we have also found a geometric formulation of mixed quantum states based on principal fibre bundles, momentum map, and purifications of quantum states. The bundle, which generalizes the Hopf bundle for pure states, gives in a canonical way rise to a Riemannian metric and a symplectic structure on the quantum phase space.

In this lecture, I will give a short introduction to the main ideas of geometric formulation of quantum mechanics with more emphasizing on quantum dynamics. I will also review our geometric framework for mixed quantum states with some applications such as geometric phase and quantum speed limit.